THE DISCRETE MODULE CATEGORY FOR THE RING OF K-THEORY OPERATIONS

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Abstract. We study the category of discrete modules over the ring of degree zero stable operations in p-local complex K-theory. We show that the $K_{(p)}$-homology of any space or spectrum is such a module, and that this category is isomorphic to a category defined by Bousfield and used in his work on the $K_{(p)}$-local stable homotopy category [2]. We also provide an alternative characterisation of discrete modules as locally finitely generated modules.

1. Introduction

An explicit description of the topological ring A of degree zero stable operations in p-local complex K-theory was given in [5]. Here we consider the category of 'discrete modules' over this ring. We focus attention on *discrete* modules because the $K_{(p)}$-homology of any space or spectrum is such a module; see Proposition 2.6.

In Section 2 we recall some results from [5] about the ring A, define discrete A-modules, and show how the category $\mathcal{D}A$ of such modules is easily seen to be a cocomplete abelian category.

Our first main result comes in Section 3, where we provide an interesting alternative characterisation of discrete A-modules: an A-module is discrete if and only if it is locally finitely generated. We also show that the category $\mathcal{D}A$ is isomorphic to the category of comodules over the coalgebra $K_0(K)_{(p)}$ of which A is the dual.

The category $\mathcal{D}A$ arose in disguised form in [2]. There Bousfield introduced a certain category $\mathcal{A}(p)$ as the first step in his investigation of the $K_{(p)}$-local stable homotopy category. In Section 4 we recall Bousfield's (rather elaborate) definition, and we prove that his category is isomorphic to the category of discrete A-modules.

This allows us to simplify and clarify in Section 5 some constructions in Bousfield's work. In particular, there is a right adjoint to the forgetful functor from $\mathcal{A}(p)$ to the category of $\mathbb{Z}_{(p)}$-modules. For Bousfield, this functor has to be constructed in an ad hoc fashion, separating cases. In our context, it is revealed as simply a continuous Hom

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functor. We give a construction in this language of a four-term exact sequence involving the right adjoint.

Bousfield's aim was to give an algebraic description of the $K_{(p)}$-local stable homotopy category. He succeeded at the level of objects, and in Section 6 we translate his main result into our language of discrete A-modules.

Of course, p-local K-theory splits as a sum of copies of the Adams summand. We have chosen to write the main body of this paper in the non-split context, but very similar results hold in the split setting. We record these in an appendix.

We note that a full algebraic description of the $K_{(p)}$-local stable homotopy category has been given by Franke [6]. The interested reader may wish to consult [12, 8]. In a different direction, we note the further work of Bousfield, building a unified version of K-theory in order to combine information from different primes [3].

Throughout this paper p will denote an odd prime.

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2. Discrete A-modules

Let p be an odd prime and let $A = K^0_{(p)}(K_{(p)})$ be the ring of degree zero stable operations in p-local complex K-theory. This ring can be described as follows; see [5]. Choose an integer q that is primitive modulo p^2, and let $\Psi^q \in A$ be the corresponding Adams operation. Let

$$q_i = q^{(-1)^i \lfloor i/2 \rfloor}, \tag{2.1}$$

and define polynomials $\Theta_n(X)$, for each integer $n \geqslant 0$, by $\Theta_n(X) = \prod_{i=1}^n (X - q_i)$. Then let the operation $\Phi_n \in A$ be given by $\Phi_n = \Theta_n(\Psi^q)$. For example, $\Phi_4 = (\Psi^q - 1)(\Psi^q - q)(\Psi^q - q^{-1})(\Psi^q - q^2)$. These operations have been chosen so that any infinite sum $\sum_{n\geqslant 0} a_n \Phi_n$, with coefficients a_n in the p-local integers $\mathbb{Z}_{(p)}$, converges. The following theorem says that any operation can be written uniquely in this form.

Theorem 2.2. [5, Theorem 6.2] *The elements of A can be expressed uniquely as infinite sums $\sum_{n\geqslant 0} a_n \Phi_n$, where $a_n \in \mathbb{Z}_{(p)}$.* □

For each $m \geq 0$, we define

$$A_m = \left\{ \sum_{n \geqslant m} a_n \Phi_n : a_n \in \mathbb{Z}_{(p)} \right\} \subseteq A,$$

so that A_m is the ideal of operations which annihilate the coefficient groups $\pi_{2i}(K_{(p)})$ for $-m/2 < i < (m+1)/2$, and thus it does not

depend on the choice of primitive element q. We obtain a decreasing filtration

$$A = A_0 \supset A_1 \supset \cdots \supset A_m \supset A_{m+1} \supset \cdots .$$

We use this filtration to give the ring A a topology in the standard way (see, for example, Chapter 9 of [11]): the open sets are unions of sets of the form $y + A_m$, where $y \in A$ and $m \geqslant 0$. We note that A is complete with respect to this topology. Indeed, another way to view the topological ring A is as the completion of the polynomial ring $\mathbb{Z}_{(p)}[\Psi^q]$ with respect to the filtration by the principal ideals $\Phi_m \mathbb{Z}_{(p)}[\Psi^q] = \mathbb{Z}_{(p)}[\Psi^q] \cap A_m$. (By way of warning, this filtration is not multiplicative in the sense of [11], and, since A is not Noetherian [5, Theorem 6.10], it is not the completion of a polynomial ring with respect to *any* multiplicative filtration.)

Definition 2.3. A *discrete A-module M* is an A-module such that the action map

$$A \times M \to M$$

is continuous with respect to the discrete topology on M and the resulting product topology on $A \times M$.

In practice we use the following criterion to recognise discrete modules.

Lemma 2.4. *An A-module M is discrete if and only if for each $x \in M$, there is some n such that $A_n x = 0$.*

Proof. Fixing $x \in M$, the map sending $\alpha \in A$ to $(\alpha, x) \in A \times M$ is continuous. Thus if M is discrete, the map $\alpha \mapsto \alpha x \in M$ is continuous, which implies that its kernel contains A_n for some n.

Suppose now that for each $x \in M$, there exists n such that $A_n x = 0$. If $x, y \in M$ are such that $\alpha x = y$, and $A_n x = 0$, then $(\alpha + A_n)x = y$ so that $(\alpha + A_n) \times \{x\}$ is an open neighbourhood of (α, x) in the preimage under the action map of $\{y\}$. This shows that M is a discrete module. $\square$

Remark 2.5. For $n \geqslant 1$, the principal ideal $\Phi_n A$ is *strictly* contained in A_n; see Proposition 3.3. Thus, if an A-module M has the property that for each $x \in M$ there is some $n \geqslant 0$ with $\Phi_n x = 0$, then it does not follow that M is a discrete A-module.

For example, the A-module $A/\Phi_1 A$ is not discrete, although every element is annihilated by Φ_1. For suppose the element $1 + \Phi_1 A \in A/\Phi_1 A$ were annihilated by the ideal A_n, i.e., $A_n \subseteq \Phi_1 A$. Then $A/\Phi_1 A$ would be finitely generated over $\mathbb{Z}_{(p)}$, being a quotient of A/A_n which has rank n over $\mathbb{Z}_{(p)}$. However, $A/\Phi_1 A$ contains the submodule $A_1/\Phi_1 A$ which is isomorphic to $\mathbb{Z}_p/\mathbb{Z}_{(p)}$, where $\mathbb{Z}_p$ denotes the p-adic integers. The proof of this is analogous to that of the corresponding result for connective K-theory which is given in [5, Corollary 3.7].

We will see in Section 3 that if Ax is finitely generated over $\mathbb{Z}_{(p)}$, then $\Phi_n x = 0$ does imply $A_n x = 0$.

The motivating example for the concept of a discrete A-module is the p-local K-homology of a spectrum.

Proposition 2.6. *The degree zero p-local K-homology $K_0(X;\mathbb{Z}_{(p)})$ of a spectrum X is a discrete A-module with the action map given by*

$$A \otimes K_0(X;\mathbb{Z}_{(p)}) \to A \otimes K_0(K)_{(p)} \otimes K_0(X;\mathbb{Z}_{(p)}) \to K_0(X;\mathbb{Z}_{(p)}),$$

in which $K_0(K)_{(p)}$ denotes the ring of degree zero cooperations in p-local K-theory, the first map arises from the coaction map, and the second comes from the Kronecker pairing.

Proof. Consider the Kronecker pairing $\langle -, - \rangle : A \otimes K_0(K)_{(p)} \to \mathbb{Z}_{(p)}$. We claim that for each element $f \in K_0(K)_{(p)}$, there is some m such that $\langle A_m, f\rangle = 0$. Indeed this is clear, since $A = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(K_0(K)_{(p)}, \mathbb{Z}_{(p)})$, with the Φ_n being the dual basis to a particular $\mathbb{Z}_{(p)}$-basis of $K_0(K)_{(p)}$; see [5, Theorem 6.2].

Now let X be a spectrum, and let $x \in K_0(X;\mathbb{Z}_{(p)})$. Under the coaction $K_0(X;\mathbb{Z}_{(p)}) \to K_0(K)_{(p)} \otimes K_0(X;\mathbb{Z}_{(p)})$, the image of x is a finite sum $\sum_{i=1}^k f_i \otimes x_i$ for some elements $f_i \in K_0(K)_{(p)}$ and $x_i \in K_0(X;\mathbb{Z}_{(p)})$. By the preceding paragraph, for each f_i, there is some m_i such that $\langle A_{m_i}, f_i\rangle = 0$. Hence, if $m = \max\{m_1, \dots, m_k\}$, we have $\langle A_m, f_i\rangle = 0$ for all i. Thus $A_m x = \sum_{i=1}^k \langle A_m, f_i\rangle x_i = 0$. □

If an A-module is a discrete A-module, then the A-action is determined by the action of Φ_1, or, equivalently, by the action of $\Psi^q = 1+\Phi_1$.

Lemma 2.7. *If M and N are discrete A-modules, and $f : M \to N$ is a $\mathbb{Z}_{(p)}$-homomorphism which commutes with the action of Ψ^q, then f is a homomorphism of A-modules.*

Proof. Since $\Phi_i = \Theta_i(\Psi^q)$, the map f must also commute with Φ_i and hence with all finite linear combinations $\sum_{i=0}^k a_i\Phi_i$. Now let $\alpha = \sum_{i\geqslant 0} a_i\Phi_i \in A$ and $x \in M$. Since M and N are discrete A-modules, for some n, we have $A_n x = 0$ and $A_n f(x) = 0$. Then

$$f\left(\sum_{i\geqslant 0} a_i\Phi_i x\right) = f\left(\sum_{i=0}^{n-1} a_i\Phi_i x\right) = \sum_{i=0}^{n-1} a_i\Phi_i f(x) = \sum_{i\geqslant 0} a_i\Phi_i f(x).$$

Hence f commutes with all elements of A, i.e., it is an A-homomorphism. □

We end this section by making some remarks about the category of discrete A-modules.

Definition 2.8. We define *the category of discrete A-modules $\mathcal{D}A$* to be the full subcategory of the category of A-modules whose objects are discrete A-modules.

Lemma 2.9. *The category $\mathcal{D}A$ is closed under submodules and quotients.*

Proof. It is clear that a submodule of a discrete A-module is discrete.

Let M be a discrete A-module and suppose that N is an A-submodule of M. Consider the quotient A-module M/N. For $x \in M$, there is some n such that $A_n x = 0$. Then $A_n(x+N) = 0$, and so M/N is a discrete A-module. □

Lemma 2.10. *The category $\mathcal{D}A$ has arbitrary direct sums.*

Proof. Let M_i be a discrete A-module for each i in some indexing set $\mathcal{I}$. Each element of $\oplus_{i\in\mathcal{I}} M_i$ is a finite sum $x_1 + \cdots + x_k$, where each x_j belongs to some M_i. For each j, there is some n_j such that $A_{n_j} x_j = 0$. Hence $A_n(x_1+\cdots+x_k) = 0$ if $n \geqslant \max\{n_1, \ldots, n_k\}$. So the direct sum $\oplus_{i\in\mathcal{I}} M_i$ is discrete. □

Corollary 2.11. *$\mathcal{D}A$ is a cocomplete abelian category.*

Proof. That $\mathcal{D}A$ is abelian follows directly from Lemmas 2.9 and 2.10 and the fact that A-modules form an abelian category. An abelian category with arbitrary direct sums is cocomplete; see, for example, [13, Proposition 2.6.8]. □

When combined with Theorem 4.6 below, the following result corresponds to 10.5 of [2].

Proposition 2.12. *$\mathcal{D}A$ is isomorphic to the category of $K_0(K)_{(p)}$-comodules.*

Proof. Let $G_n(w) = q^{n\lfloor n/2\rfloor} F_n(w) \in K_0(K)_{(p)}$, where $F_n(w)$ is as defined in the proof of Theorem 6.2 in [5]. That proof shows that the Φ_n are dual to the $G_n(w)$, i.e., $\langle \Phi_n, G_i(w)\rangle = \delta_{ni}$.

If M is a discrete A-module, define $\varphi_M : M \to K_0(K)_{(p)} \otimes M$ by $\varphi_M(x) = \sum_{n\geqslant 0} G_n(w) \otimes \Phi_n x$, in which only finitely many terms are nonzero because M is discrete. It is easily checked that this makes M into a $K_0(K)_{(p)}$-comodule and that an A-module homomorphism between discrete A-modules is also a $K_0(K)_{(p)}$-comodule homomorphism.

If M is a $K_0(K)_{(p)}$-comodule with coaction $\varphi_M : M \to K_0(K)_{(p)} \otimes M$, then, just as in the proof of Proposition 2.6, M is a discrete A-module with action given by

$$A \otimes M \xrightarrow{1\otimes\varphi_M} A \otimes K_0(K)_{(p)} \otimes M \xrightarrow{\langle -,-\rangle\otimes 1} M.$$

It is clear that this construction is functorial.

It is routine to check that the two constructions are mutually inverse. □

3. Locally Finitely Generated A-modules

Definition 3.1. An A-module M is *locally finitely generated* if, for every $x \in M$, the submodule Ax is finitely generated over $\mathbb{Z}_{(p)}$.

This section is devoted to proving the following theorem.

Theorem 3.2. *An A-module M is discrete if and only if it is locally finitely generated.*

We need to prove first a number of preliminary results.

Proposition 3.3. *For all $n \geqslant 1$, the quotient $A_n/\Phi_n A$ is a rational vector space.*

Proof. The result will follow if we can show that $A_n/\Phi_n A$ is divisible and torsion-free. We thus need to show:

(1) For any $\alpha \in A_n$, there exists $\beta \in A$ such that $\Phi_n\beta - \alpha \in pA_n$;
(2) if $\alpha \in A_n$ and $p\alpha \in \Phi_n A$, then $\alpha \in \Phi_n A$.

Recall from Proposition 6.8 of [5] that

$$\Phi_n\Phi_j = \sum_{k=\max(j,n)}^{j+n} c_{j,n}^k \Phi_k$$

for certain coefficients $c_{j,n}^k \in \mathbb{Z}_{(p)}$ (denoted $A_{j,n}^k$ in [5]).

Suppose $\alpha = \sum_{k \geqslant n} a_k \Phi_k \in A_n$, then $\beta = \sum_{j \geqslant 0} b_j \Phi_j \in A$ will satisfy $\Phi_n\beta - \alpha \in pA_n$ if and only if the congruences

$$\sum_{j=k-n}^{k} c_{j,n}^k b_j \equiv a_k \mod p \quad (k \geqslant n) \tag{3.4}$$

can be solved for $(b_j)_{j\geqslant 0}$. We will verify (1) by showing that these congruences *can* always be solved, and (2) by showing that the solution is unique modulo p.

It follows from Propositions A.2 and A.4 of [5] that, with q_k as defined in (2.1),

$$c_{j,n}^k = (q_{k+1} - q_n)c_{j,n-1}^k + c_{j,n-1}^{k-1}, \tag{3.5}$$

where $c_{j,n}^k = 0$ unless $j, n \leqslant k \leqslant j+n$, and $c_{j,n}^k = 1$ if $k = j+n$. We claim that for any integer $s \geqslant 1$,

$$c_{j,n}^k \equiv 0 \mod p \quad \text{for } (2p-2)s \leqslant j \leqslant k < (2p-2)s + n. \tag{3.6}$$

This follows by induction on n from (3.5) and the periodicity of the sequence $(q_i)_{i\geqslant 1}$ modulo p: $q_i \equiv q_{i+2p-2} \mod p$.

It follows from (3.6) that for $k = (2p-2)s+n-1$ the congruence (3.4) is $b_{(2p-2)s-1} \equiv a_{(2p-2)s+n-1} \mod p$, and that for $k < (2p-2)s+n-1$ the congruences have the form

$$b_{k-n} + \big(\text{terms involving } b_j \text{ for } k-n < j < (2p-2)s\big) \equiv a_k \mod p.$$

It is now clear that the congruences (3.4) have a unique solution modulo p for the b_j. $\square$

Corollary 3.7. *If M is a locally finitely generated A-module and $x \in M$ satisfies $\Phi_n x = 0$ for some $n \geqslant 0$, then $A_n x = 0$.*

Proof. By hypothesis the map $A_n \to Ax$ sending α to αx factors through the $\mathbb{Q}$-module $A_n/\Phi_n A$. Hence, since Ax is a finitely generated $\mathbb{Z}_{(p)}$-module, the map must be zero. $\square$

We now need to show that, for an element x in a locally finitely generated A-module M, we have $\Phi_n x = 0$ for some $n \geqslant 0$. The next results establish this, first for $\mathbb{Z}_{(p)}$-torsion modules, then for $\mathbb{Z}_{(p)}$-free modules and finally in the general case.

Proposition 3.8. *If M is a finite A-module, then $\Phi_n M = 0$ for some n.*

Proof. Let $x \in M$ be non-zero, and define

$$I = \{\, f(X) \in \mathbb{Z}_{(p)}[X] : f(\Psi^q)x = 0 \,\},$$

which is clearly an ideal of $\mathbb{Z}_{(p)}[X]$.

Let $f(X)$ be any element of I such that its reduction $\bar{f}(X) \in \mathbb{F}_p[X]$ is not zero. Since the elements $\Psi^{q^r}x$ for $r \geqslant 0$ cannot be distinct, we may, for example, take an $f(X)$ of the form $X^{r_1} - X^{r_2}$ with $r_1 \neq r_2$.

Suppose that

$$\bar{f}(X) = \bar{g}(X) \prod_{k=1}^{p-1} (X-k)^{e_k} \qquad (e_k \geqslant 0),$$

where $\bar{g}(X)$ has no roots in $\mathbb{F}_p^{\times}$, and thus we can write

$$f(X) = g(X) \prod_{k=1}^{p-1} (X-k)^{e_k} + ph(X),$$

for some $g(X), h(X) \in \mathbb{Z}_{(p)}[X]$, where $g(k) \in \mathbb{Z}_{(p)}^{\times}$ for $k = 1, 2, \ldots, p-1$.

We recall now from [10] and [5, §6] that if $\varphi(X) \in \mathbb{Z}_{(p)}[X]$, the element $\varphi(\Psi^q)$ is a unit in A if and only if $\varphi(q_i)$ is a unit in $\mathbb{Z}_{(p)}$ for all $i \geqslant 1$. Since the q_i take the values $1, 2, \ldots, p-1$ modulo p, it follows that $g(\Psi^q)$ is a unit in A. Since $f(\Psi^q)$ cannot be a unit, it follows that $e_k > 0$ for at least one value of k.

We have

$$g(\Psi^q) \prod_{k=1}^{p-1} (\Psi^q - k)^{e_k} x = -ph(\Psi^q)x,$$

and thus

$$\prod_{k=1}^{p-1} (\Psi^q - k)^{e_k} x = p\alpha x,$$

where $\alpha = -g(\Psi^q)^{-1} h(\Psi^q) \in A$.

As M is finite, there is some s such that $p^s x = 0$, so that

$$\prod_{k=1}^{p-1}(\Psi^q - k)^{e_k s} x = p^s \alpha^s x = 0.$$

But it is clear that $\prod_{k=1}^{p-1}(X-k)^{e_k s}$ is a factor modulo p^s of $\Theta_n(X)$ for sufficiently large n, and thus $\Phi_n x = 0$ for such n.

By choosing the maximum such n over all non-zero $x \in M$, we have $\Phi_n M = 0$. □

Proposition 3.9. *If M is an A-module which is free of finite rank over $\mathbb{Z}_{(p)}$, then $\Phi_n M = 0$ for some n.*

Proof. Let $x \in M$. The action map $\eta_x : A \to M$ given by $\eta_x(\alpha) = \alpha x$ is a homomorphism of A-modules. In particular, it is a homomorphism of abelian groups. By [7, Theorem 95.3], the target is a slender group, so there is some m such that $\eta_x(\Phi_m) = 0$, i.e., $\Phi_m x = 0$.

If $x_1, \ldots, x_r$ is a $\mathbb{Z}_{(p)}$-basis of M and $\Phi_{m_i} x_i = 0$ for $i = 1, \ldots, r$, then $\Phi_n M = 0$ where $n = \max\{m_i : 1 \leqslant i \leqslant r\}$. □

Since A is not an integral domain, we must exercise some care with quotients. However, if $n > m$ the polynomial $\Theta_m(X)$ is a factor of $\Theta_n(X)$, so we may let Φ_n/Φ_m denote the value of the polynomial $\Theta_n(X)/\Theta_m(X)$ at Ψ^q.

Lemma 3.10. *If $n > m$ and $n - m$ is divisible by $2p-2$, then*

$$\frac{\Phi_n}{\Phi_m} \equiv \Phi_{n-m} \mod p^{1+\nu_p(n-m)}.$$

Proof. It is easy to verify that whenever $n > m > 0$,

$$\frac{\Phi_n}{\Phi_m} = \frac{\Phi_{n-1}}{\Phi_{m-1}} + (q_m - q_n)\frac{\Phi_{n-1}}{\Phi_m}.$$

Expanding the first term on the right in the same way, and repeating this process, leads to the equation

$$\frac{\Phi_n}{\Phi_m} = \Phi_{n-m} + \sum_{i=0}^{m-1}(q_{m-i} - q_{n-i})\frac{\Phi_{n-i-1}}{\Phi_{m-i}}.$$

If $n - m = 2k$, then $q_{m-i} - q_{n-i}$ is divisible by $q^k - 1$ for all i. Moreover, if k is divisible by $(p-1)p^{r-1}$, then $q^k - 1 \equiv 0 \mod p^r$. □

Proof of Theorem 3.2. First suppose that M is a discrete A-module. Then, for $x \in M$, there is some n such that $Ax = (A/A_n)x$. But A/A_n is free of finite rank over $\mathbb{Z}_{(p)}$, so M is locally finitely generated.

Now suppose that M is locally finitely generated. By Corollary 3.7, it is enough to show that for each $x \in M$ there is some n such that $\Phi_n x = 0$.

Let $x \in M$, and write $N = Ax$. By hypothesis, N is finitely generated over $\mathbb{Z}_{(p)}$. Let $T \subseteq N$ be the A-submodule of N consisting of

the $\mathbb{Z}_{(p)}$-torsion elements. So there is some s such that $p^sT = 0$. The quotient N/T is an A-module which is free of finite rank over $\mathbb{Z}_{(p)}$. By Proposition 3.9, there is some k such that $\Phi_k(N/T) = 0$, and so $\Phi_k x \in T$. Then, by Proposition 3.8, there is some r such that $\Phi_r\Phi_k x = 0$.

By increasing r if necessary, we may arrange that r is divisible by $(2p-2)p^s$, so that, by Lemma 3.10, we have

$$\frac{\Phi_{r+k}}{\Phi_k} = \Phi_r + p^s\theta,$$

for some $\theta \in A$. Hence

$$\Phi_{r+k}x = \frac{\Phi_{r+k}}{\Phi_k}\Phi_k x = \Phi_r\Phi_k x + p^s\theta\Phi_k x = 0 + \theta(p^s\Phi_k x) = 0.$$

□

4. Bousfield's Category of K-theory modules

In this section we relate the category $\mathcal{D}A$ of discrete A-modules to a category considered by Bousfield in his work on the $K_{(p)}$-local stable homotopy category.

Let $R = \mathbb{Z}_{(p)}[\mathbb{Z}_{(p)}^\times]$ be the group-ring of the multiplicative group of units in $\mathbb{Z}_{(p)}$, with coefficients in $\mathbb{Z}_{(p)}$ itself. For clarity, as well as to reflect the topological applications, we write $\Psi^j \in R$ for the element $j \in \mathbb{Z}_{(p)}^\times$. Hence elements of R are finite $\mathbb{Z}_{(p)}$-linear combinations of the Ψ^j.

Definition 4.1. [2] *Bousfield's category* $\mathcal{A}(p)$ is the full subcategory of the category of R-modules whose objects M satisfy the following conditions. For each $x \in M$,

(a) the submodule $Rx \subseteq M$ is finitely generated over $\mathbb{Z}_{(p)}$,
(b) for each $j \in \mathbb{Z}_{(p)}^\times$, Ψ^j acts on $Rx \otimes \mathbb{Q}$ by a diagonalisable matrix whose eigenvalues are integer powers of j,
(c) for each $m \geqslant 1$, the action of $\mathbb{Z}_{(p)}^\times$ on Rx/p^mRx factors through the quotient homomorphism $\mathbb{Z}_{(p)}^\times \to (\mathbb{Z}/p^k\mathbb{Z})^\times$ for sufficiently large k.

We call the objects of this category *Bousfield modules.*

If M is a Bousfield module which is finitely generated over $\mathbb{Z}_{(p)}$, then condition (a) holds automatically, and the rational diagonalisability and p-adic continuity conditions of (b) and (c) hold globally for M, as well as for each submodule $Rx \subseteq M$.

Note that $R \subset A$, so an A-module can be considered as an R-module by restricting the action. (An explicit formula expressing each Ψ^j, for $j \in \mathbb{Z}_{(p)}^\times$, in terms of the Φ_n is given in [5, Proposition 6.6].)

Theorem 4.2. *If M is a discrete A-module, then M is a Bousfield module with the R-action given by the inclusion $R \subset A$.*

Proof. Suppose $A_n x = 0$, then it is clear that $Rx = Ax = (A/A_n)x$, which is finitely generated over $\mathbb{Z}_{(p)}$. The matrix representing Ψ^q on $Rx \otimes \mathbb{Q}$ is annihilated by the polynomial $\Theta_n(X)$, and thus its minimal polynomial is a factor of $\Theta_n(X)$. Since $\Theta_n(X)$ has distinct rational roots, the matrix can be diagonalised over $\mathbb{Q}$. The eigenvalues are roots of $\Theta_n(X)$, which are integer powers of q.

Proposition 6.6 of [5] shows that

$$\Psi^j = \sum_{i \geqslant 0} g_i(j)\Phi_i,$$

if $j \in \mathbb{Z}_{(p)}^\times$, where $g_i(w)$ is a certain Laurent polynomial (given explicitly in [5]) satisfying $g_i(j) \in \mathbb{Z}_{(p)}$ for all $j \in \mathbb{Z}_{(p)}^\times$. Since $A_n x = 0$, this shows that Ψ^j acts on Rx as the polynomial in Ψ^q

$$P_j(\Psi^q) = \sum_{i=0}^{n-1} g_i(j)\Theta_i(\Psi^q).$$

It follows that the eigenvalues of the matrix of the action of Ψ^j on $Rx \otimes \mathbb{Q}$ are $P_j(q^r)$, where q^r is an eigenvalue of the action of Ψ^q. By considering the action on the coefficient group $\pi_{2r}(K_{(p)})$, we see that $P_j(q^r) = j^r$. Hence condition (b) of Definition 4.1 is satisfied.

The Laurent polynomials $g_i(j)$ are uniformly p-adically continuous functions of $j \in \mathbb{Z}_{(p)}^\times$, i.e., for each $m \geqslant 1$ there is an integer K_i such that $g_i(j) \equiv g_i(j + p^k a) \mod p^m$ whenever $k \geqslant K_i$. Hence $\Psi^j x \equiv \Psi^{j+p^k a} x \mod p^m$ for $k \geqslant \max\{K_0, K_1, \dots, K_{n-1}\}$. This shows that condition (c) of Definition 4.1 holds. □

Since each Φ_n is a polynomial in Ψ^q, any finite linear combination of the Φ_n can be considered as an element of R. In order to show that a Bousfield module can be given the structure of an A-module, we need to specify how an infinite sum $\sum_{n \geqslant 0} a_n \Phi_n$ acts. We need a preliminary lemma.

Lemma 4.3. *Let M be a Bousfield module and $x \in M$. There is some $k \geqslant 1$ such that $\Phi_n x \equiv 0 \mod p$ for all $n \geqslant p^k(p-1)$.*

Proof. Proposition 6.5 of [5] gives an explicit formula for the expansion of $\Phi_{p^k(p-1)}$ as a finite $\mathbb{Z}_{(p)}$-linear combination of the Ψ^{q^j}. The coefficient of Ψ^{q^j} has as a factor the q-binomial coefficient $\left[{p^k(p-1) \atop j}\right]$, which is divisible by p for $0 < j < p^k(p-1)$. Thus

$$\begin{aligned} \Phi_{p^k(p-1)} &\equiv \Psi^{q^{p^k(p-1)}} + q^{p^k(p-1)/2} \mod p \\ &\equiv \Psi^{q^{p^k(p-1)}} - 1 \mod p. \end{aligned}$$

Now condition (c) of Definition 4.1 ensures that $(\Psi^{q^{p^k(p-1)}} - 1)x \equiv 0 \mod p$ for sufficiently large k. Thus $\Phi_{p^k(p-1)}x \equiv 0 \mod p$, and consequently $\Phi_n x \equiv 0 \mod p$ for all $n \geqslant p^k(p-1)$. □

Theorem 4.4. *If the R-module M is a Bousfield module, then the R-action extends uniquely to an A-action in such a way as to make M a discrete A-module.*

Proof. Let $x \in M$. By condition (b) of Definition 4.1, the minimal polynomial of the action of Ψ^q on $Rx \otimes \mathbb{Q}$ has the form $\prod_{i=1}^{t}(X - q^{k_i})$, where $k_i \in \mathbb{Z}$. For sufficiently large m, this polynomial is a factor of $\Theta_m(X)$, so that $\Phi_m x = 0$ in $Rx \otimes \mathbb{Q}$. This means that $\Phi_m x \in T$, the $\mathbb{Z}_{(p)}$-torsion submodule of Rx. As T is finitely generated, there is some exponent e such that $p^e T = 0$.

Let k be as in Lemma 4.3, and let $n = (2p-2)p^r$, where $r \geqslant k$. Then $\Phi_n x \equiv 0 \mod p$, and so $\Phi_n^{\ell} x \equiv 0 \mod p^{\ell}$ for any $\ell \geqslant 1$.

On the other hand, iterating Lemma 3.10 shows that $\Phi_{2^s n} \equiv \Phi_n^{2^s} \mod p^{1+r}$. It is thus clear that by choosing r and s sufficiently large we may ensure firstly that $\Phi_{2^s n} x \in T$ and then that $\Phi_{2^s n} x \equiv 0 \mod p^e$, which means that $\Phi_{2^s n} x = 0$.

There is thus some integer N such that $\Phi_N x = 0$. Given $\alpha = \sum_{k \geqslant 0} a_k \Phi_k \in A$, let $\alpha x = \sum_{k=0}^{N-1} a_k \Phi_k x$. It is clear that this gives M the structure of a discrete A-module. The uniqueness of this structure follows from Lemma 2.7. □

Remark 4.5. We have chosen to give a direct algebraic proof. This can, of course, be replaced by appealing to Bousfield's topological results: the non-split analogue of [2, Proposition 8.7] shows that any object in $\mathcal{A}(p)$ can be expressed as $K_0(X; \mathbb{Z}_{(p)})$ for some spectrum X. Proposition 2.6 shows that $K_0(X; \mathbb{Z}_{(p)})$ is a discrete A-module.

Assembling the main results of this section, we have proved the following.

Theorem 4.6. *Bousfield's category $\mathcal{A}(p)$ is isomorphic to the category $\mathcal{D}A$ of discrete A-modules.* □

5. Cofree Objects and Injective Resolutions

To illustrate the utility of our point of view, we consider cofree objects. Bousfield showed the existence of a right adjoint functor U to the forgetful functor from his category $\mathcal{A}(p)$ to the category of $\mathbb{Z}_{(p)}$-modules. To do this, he had to give different descriptions in two special cases and then deduce the existence of such a functor in the general case without constructing it explicitly.

In our context, the functor U is just a continuous Hom functor, right adjoint to the forgetful functor from $\mathcal{D}A$ to $\mathbb{Z}_{(p)}$-modules. So, not only do we not need to treat separate cases, but our uniform description

is conceptually simple and fits into a standard framework for module categories.

If S is a unital, commutative algebra over a commutative ring R with 1, then there is a right adjoint to the forgetful functor from the category of S-modules to the category of R-modules, given by $\mathrm{Hom}_R(S,-)$; see [13, Lemma 2.3.8]. For an R-module L, $\mathrm{Hom}_R(S,L)$ is an S-module via $(sf)(t) = f(ts)$ for $s,t \in S$.

We will need to modify this construction since $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,L)$ need not be a *discrete* A-module.

Example 5.1. The A-module $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,\mathbb{Q})$ is not discrete. To see this, note that, since $\mathbb{Q}$ is not slender [7, Section 94], there exists $f \in \mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,\mathbb{Q})$ such that $f(\Phi_n) \neq 0$ for all n. Then $(\Phi_n f)(1) = f(\Phi_n) \neq 0$ for all n, and so there is no n such that $A_n f = 0$.

Notice that we can make $\mathbb{Q}$ into a discrete A-module via $A \xrightarrow{\varepsilon} \mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$, where $A \xrightarrow{\varepsilon} \mathbb{Z}_{(p)}$ is the augmentation given by $\varepsilon\big(\sum_{k\geqslant 0} a_k\Phi_k\big) = a_0$. Thus $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,L)$ need not be a discrete A-module, even when L is such a module.

Definition 5.2. If L is a $\mathbb{Z}_{(p)}$-module, let $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A,L)$ denote the A-submodule of $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,L)$ consisting of the homomorphisms which are continuous with respect to the filtration topology on A and the discrete topology on L.

Note that a homomorphism $A \to L$ is continuous if and only if its kernel contains A_n for some n. Example 5.1 shows that $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A,L)$ may be a proper submodule of $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,L)$ even if L is a discrete A-module.

Proposition 5.3. *The functor U from $\mathbb{Z}_{(p)}$-modules to the category $\mathcal{D}A$ of discrete A-modules given by $UL = \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A,L)$ is right adjoint to the forgetful functor.*

Proof. For any A-module N, we define the discrete heart of N to be the largest A-submodule which is a discrete A-module:

$$N^{\mathrm{disc}} = \{\, x \in N : A_n x = 0 \text{ for some } n \,\}.$$

It is easy to show that the 'discrete heart functor' is right adjoint to the forgetful functor from discrete A-modules to A-modules, hence $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,-)^{\mathrm{disc}}$ is right adjoint to the forgetful functor from discrete A-modules to $\mathbb{Z}_{(p)}$-modules. The proof is completed by observing that $\mathrm{Hom}_{\mathbb{Z}_{(p)}}(A,L)^{\mathrm{disc}} = \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A,L)$ for any $\mathbb{Z}_{(p)}$-module L. □

Proposition 5.4. *For any $\mathbb{Z}_{(p)}$-module L, there is a natural isomorphism of A-modules $UL \cong K_0(K)_{(p)} \otimes L$, where $K_0(K)_{(p)} \otimes L$ is an A-module via the A-module structure on $K_0(K)_{(p)}$.*

Proof. The map $K_0(K)_{(p)} \to \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, \mathbb{Z}_{(p)})$ which sends x to the homomorphism $\alpha \mapsto \varepsilon(\alpha x)$, where $\varepsilon : K_0(K)_{(p)} \to \mathbb{Z}_{(p)}$ is the augmentation, is a homomorphism of A-modules. It is an isomorphism since A is the $\mathbb{Z}_{(p)}$-dual of the free $\mathbb{Z}_{(p)}$-module $K_0(K)_{(p)}$; see [5]. Hence the result holds for the case $L = \mathbb{Z}_{(p)}$.

For a general $\mathbb{Z}_{(p)}$-module L the natural A-module homomorphism $U\mathbb{Z}_{(p)} \otimes L \to \mathrm{Hom}_{\mathbb{Z}_{(p)}}(A, L)$ maps into $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, L) = UL$. We can define the inverse A-homomorphism as follows. Let $\rho_j \in U\mathbb{Z}_{(p)}$ be given by $\rho_j(\Phi_k) = \delta_{jk}$. If $\sigma \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, L)$ and $\sigma(A_n) = 0$, we send σ to $\sum_{j=0}^{n-1} \rho_j \otimes \sigma(\Phi_j) \in U\mathbb{Z}_{(p)} \otimes L$. $\square$

Theorem 5.5. *The functor U is exact, and preserves direct sums and direct limits.*

Proof. As a right adjoint, U is left exact. On the other hand, if $f : L_1 \to L_2$ is a $\mathbb{Z}_{(p)}$-module epimorphism, then $Uf : UL_1 \to UL_2$ is an epimorphism of discrete A-modules, for if $g \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, L_2)$, then we can define $h \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, L_1)$ with $g = Uf(h)$ as follows. For each $n \geqslant 0$, we can find some $y_n \in L_1$ such that $f(y_n) = g(\Phi_n)$, and, since $g(\Phi_n) = 0$ for $n \gg 0$, we may choose y_n to be 0 for $n \gg 0$. Thus we can define $h : A \to L_1$ by $h(\sum_{n\geqslant 0} a_n\Phi_n) = \sum_{n\geqslant 0} a_n y_n$.

That U preserves direct sums is immediate from the definition, and an exact functor which preserves direct sums automatically preserves direct limits. $\square$

Corollary 5.6. *If D is an injective $\mathbb{Z}_{(p)}$-module, then UD is injective. Hence $\mathcal{D}A$ has enough injectives.*

Proof. The injectivity of UD follows from adjointness. As each $\mathbb{Z}_{(p)}$-module can be embedded in an injective $\mathbb{Z}_{(p)}$-module D, so any discrete A-module can be embedded as a $\mathbb{Z}_{(p)}$-module in such a D. Then, using adjointness and left-exactness of U, any discrete A-module can be embedded as an A-module in some UD. $\square$

Bousfield introduced in [2, (7.4)] a four-term exact sequence which underlies the fact, due to Adams and Baird [1], that all $\mathrm{Ext}^{>2}$ groups are zero. Bousfield's construction applies to the version of his category which corresponds to the split summand of K-theory; we briefly discuss this category in the Appendix. We end this section by constructing the corresponding exact sequence in $\mathcal{D}A$, using the functor U.

Theorem 5.7. *For any M in $\mathcal{D}A$, there is an exact sequence in $\mathcal{D}A$:*

$$0 \to M \xrightarrow{\alpha} UM \xrightarrow{\beta} UM \xrightarrow{\gamma} M \otimes \mathbb{Q} \to 0,$$

where UM denotes the discrete A-module obtained by applying U to the $\mathbb{Z}_{(p)}$-module underlying M (i.e., applying a forgetful functor to M before applying U).

Proof. The map α is adjoint to the identity map $M \to M$; explicitly αx maps $\theta \in A$ to θx. It is clear that α is a monomorphism. For any $x \in M$, the map $\alpha x : A \to M$ is A-linear. Moreover, any A-linear map $f : A \to M$ is determined by $f(1)$, and $f = \alpha\big(f(1)\big)$. Thus the image of α is exactly the subset $\mathrm{Hom}_A(A, M)$ of A-linear maps in $UM = \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$ (an A-linear map into a discrete A-module is automatically continuous).

The map β is given by

$$\beta f = \Psi^q \circ f - f \circ \Psi^q,$$

where $f : A \to M$. Here $\Psi^q \circ f$ uses the A-module structure of M, not that of UM, in other words, $\beta f : \theta \mapsto \Psi^q f(\theta) - f(\Psi^q \theta)$. It is straightforward to check that β is an A-homomorphism. A continuous $\mathbb{Z}_{(p)}$-homomorphism from A into a discrete A-module which commutes with Ψ^q is an A-module homomorphism, hence $\mathrm{Ker}\,\beta = \mathrm{Hom}_A(A, M) = \mathrm{Im}\,\alpha$.

We define γ as follows. Let

$$\Theta_n^{(j)}(X) = \prod_{\substack{i=1\\ i\neq j}}^{n} (X - q_i),$$

and let $\Phi_n^{(j)} = \Theta_n^{(j)}(\Psi^q) \in A$. Note that $\Phi_n^{(j)} = \Phi_n$ if $j > n$, since $\Theta_n^{(j)}(X) = \Theta_n(X)$ in that case. If $x \in M$ and $\Phi_n x = 0$, then each $\Phi_n^{(j)} x$ is an eigenvector of Ψ^q with eigenvalue q_j. Moreover we have

$$1 = \Phi_0 = \sum_{j=1}^{n} \frac{\Phi_n^{(j)}}{\Theta_n^{(j)}(q_j)} \tag{5.8}$$

in $A \otimes \mathbb{Q}$ for all $n \geqslant 1$.

If $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$, choose n such that $f(\Phi_k) = 0$ for all $k \geqslant n$ and $\Phi_n f(\Phi_k) = 0$ for $0 \leqslant k < n$. Then let

$$\gamma f = \sum_{k\geqslant 0} \sum_{j=1}^{k+1} \frac{\Phi_n^{(j)} f(\Phi_k)}{\Theta_n^{(j)}(q_j)\Theta_{k+1}^{(j)}(q_j)} \in M \otimes \mathbb{Q}.$$

Note that we may reverse the order of summation to obtain

$$\gamma f = \sum_{j=1}^{n} \frac{\Phi_n^{(j)} x_j}{\Theta_n^{(j)}(q_j)}, \tag{5.9}$$

where

$$x_j = \sum_{k=j-1}^{n-1} \frac{f(\Phi_k)}{\Theta_{k+1}^{(j)}(q_j)}.$$

In this form it is apparent that the formula for γf is independent of the choice of n. For if $n \leqslant m$ and $1 \leqslant j \leqslant m$, it follows from

$\Psi^q\Phi_n^{(j)}x_j = q_j\Phi_n^{(j)}x_j$ that

$$\frac{\Phi_m^{(j)}x_j}{\Theta_m^{(j)}(q_j)} = \frac{\Phi_n^{(j)}x_j}{\Theta_n^{(j)}(q_j)}.$$

Suppose now that $f = \beta g$, with $g(\Phi_k) = 0$ for all $k \geqslant n$ and $\Phi_n g(\Phi_k) = 0$ for $0 \leqslant k < n$. Then

$$x_j = \sum_{k=j-1}^{n-1} \frac{(\Psi^q - q_{k+1})g(\Phi_k) - g(\Phi_{k+1})}{\Theta_{k+1}^{(j)}(q_j)}.$$

But, since $\Theta_{k+1}^{(j)}(q_j) = (q_j - q_{k+1})\Theta_k^{(j)}(q_j)$ for $1 \leqslant j \leqslant k$, and $g(\Phi_n) = 0$,

$$x_j = \sum_{k=j-1}^{n-1} \frac{(\Psi^q - q_j)g(\Phi_k)}{\Theta_{k+1}^{(j)}(q_j)}.$$

It follows from (5.9) that $\gamma\beta g = 0$, since $\Phi_n^{(j)}(\Psi^q - q_j) = \Phi_n$.

To prove that $\operatorname{Ker}\gamma \subseteq \operatorname{Im}\beta$ we need first the following lemma.

Lemma 5.10. *Suppose $f \in \operatorname{Ker}\gamma$, where M is a discrete A-module and n is chosen as above. Then $\sum_{k=0}^{n-1}(\Phi_n/\Phi_{k+1})f(\Phi_k)$ is a $\mathbb{Z}_{(p)}$-torsion element of M.*

Proof. Since $\gamma f = 0$, in equation (5.9) each $\Phi_n^{(j)}x_j = 0$ in $M \otimes \mathbb{Q}$, since otherwise Ψ^q would have linearly dependent eigenvectors with distinct eigenvalues. Thus for each $j = 1, \ldots, n$,

$$0 = \sum_{k=j-1}^{n-1} \frac{\Phi_n^{(j)} f(\Phi_k)}{\Theta_{k+1}^{(j)}(q_j)}. \tag{5.11}$$

Now if $j \leqslant k+1 \leqslant n$, $\Phi_n^{(j)} = \Phi_{k+1}^{(j)}(\Phi_n/\Phi_{k+1})$ in A. Hence (5.11) becomes

$$0 = \sum_{k=j-1}^{n-1} \frac{\Phi_{k+1}^{(j)}(\Phi_n/\Phi_{k+1})f(\Phi_k)}{\Theta_{k+1}^{(j)}(q_j)}$$

in $M \otimes \mathbb{Q}$. Let d_n denote the least common multiple of the $\Theta_{k+1}^{(j)}(q_j)$ for $j \leqslant k+1 \leqslant n$. Then multiplying by d_n yields $0 = z_j$ in $M \otimes \mathbb{Q}$, where

$$z_j := \sum_{k=j-1}^{n-1} \big((d_n/\Theta_{k+1}^{(j)}(q_j))\Phi_{k+1}^{(j)}\big)(\Phi_n/\Phi_{k+1})f(\Phi_k) \in M.$$

Hence z_j is a $\mathbb{Z}_{(p)}$-torsion element in M, and so is

$$\sum_{j=1}^{n} z_j = \sum_{k=0}^{n-1}\left(\sum_{j=1}^{k+1}\bigl(d_n/\Theta_{k+1}^{(j)}(q_j)\bigr)\Phi_{k+1}^{(j)}\right)(\Phi_n/\Phi_{k+1})f(\Phi_k)$$
$$= d_n\sum_{k=0}^{n-1}(\Phi_n/\Phi_{k+1})f(\Phi_k),$$

where we use (5.8). The result follows. □

Now if $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$ and $r \geqslant 1$, define $\tilde{f}_r \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$ by

$$\tilde{f}_r(\Phi_k) = \begin{cases} \displaystyle\sum_{i=0}^{k-1}(\Phi_k/\Phi_{i+1})f(\Phi_i), & \text{if } 1 \leqslant k \leqslant r,\\ 0, & \text{if } k = 0 \text{ or } k > r.\end{cases}$$

It is a simple calculation that

$$(\beta\tilde{f}_r + f)(\Phi_k) = \begin{cases} 0, & \text{if } k < r,\\ \displaystyle\sum_{i=0}^{r}(\Phi_{r+1}/\Phi_{i+1})f(\Phi_i), & \text{if } k = r,\\ f(\Phi_k), & \text{if } k > r.\end{cases}$$

Suppose now that $f \in \operatorname{Ker}\gamma$, that $f(\Phi_k) = 0$ for all $k \geqslant n$ and $\Phi_n f(\Phi_k) = 0$ for $0 \leqslant k < n$. Then if $r > n$,

$$(\beta\tilde{f}_r + f)(\Phi_r) = (\Phi_{r+1}/\Phi_n)y,$$

where $y = \sum_{i=0}^{n-1}(\Phi_n/\Phi_{i+1})f(\Phi_i)$ is, by Lemma 5.10, a $\mathbb{Z}_{(p)}$-torsion element of M. But now Lemma 3.10 shows that for r sufficiently large $(\Phi_{r+1}/\Phi_n)y = 0$, in which case $f = \beta(-\tilde{f}_r)$. Hence we have shown that $\operatorname{Ker}\gamma \subseteq \operatorname{Im}\beta$.

It remains to show that γ maps onto $M \otimes \mathbb{Q}$. Let $x \in M$, and suppose $\Phi_n x = 0$. Letting d denote the least common multiple of the $\Theta_n^{(k)}(q_k)$ for $k = 1, \ldots, n$, define $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$ by $f(\Phi_k) = \bigl(d\Theta_{k+1}^{(k)}(q_k)/\Theta_n^{(k)}(q_k)\bigr)\Phi_n^{(k)}x$. Since $\Phi_n^{(j)}\Phi_n^{(k)}x = \Theta_n^{(j)}(q_k)\Phi_n^{(k)}x$ is zero unless $j = k$, it is a simple calculation using (5.8) that $\gamma f = dx$ in $M \otimes \mathbb{Q}$.

The proof that γ is an epimorphism is completed by showing that if $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$, there exists $g \in \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(A, M)$ such that $f - pg \in \operatorname{Ker}\gamma$.

To see this, choose a multiple m of $2p-2$ such that $f(\Phi_k) = 0$ for all $k \geqslant m$ and $\Phi_m f(\Phi_k) = 0$ for all $k \geqslant 0$. By Lemma 3.10, for each $k \geqslant m$ there is a $\theta_k \in A$ such that $(\Phi_{k+1}/\Phi_{k-m+1}) = \Phi_m + p\theta_k$. We

define g as follows

$$g(\Phi_k) = \begin{cases} \theta_k f(\Phi_{k-m}), & \text{if } k \geqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

For suitably large n, let

$$y_j = \sum_{k=j-1}^{n-1} \frac{pg(\Phi_k)}{\Theta^{(j)}_{k+1}(q_j)} = \sum_{k=j-m-1}^{n-m-1} \frac{(\Phi_{k+m+1}/\Phi_{k+1})f(\Phi_k)}{\Theta^{(j)}_{k+m+1}(q_j)},$$

so that $p\gamma g = \sum_{j=1}^{n} \Phi^{(j)}_n y_j / \Theta^{(j)}_n(q_j)$. It is now clear that $p\gamma g = \gamma f$. For

$$(\Phi_{k+m+1}/\Phi_{k+1})\Phi^{(j)}_n f(\Phi_k) = \left(\prod_{i=k+2}^{k+m+1} (q_j - q_i)\right) \Phi^{(j)}_n f(\Phi_k),$$

which is zero if $k+2 \leqslant j \leqslant k+m+1$, while

$$\Theta^{(j)}_{k+m+1}(q_j) = \Theta^{(j)}_{k+1}(q_j) \prod_{i=k+2}^{k+m+1} (q_j - q_i),$$

if $1 \leqslant j \leqslant k+1$. □

6. Bousfield's Main Result

We translate Bousfield's main result, giving an algebraic classification of $K_{(p)}$-local homotopy types, into our language of discrete A-modules. We claim no originality here, but we believe it is useful to summarise Bousfield's results in our language.

There are several steps in Bousfield's construction: we need a graded version of our discrete module category, we need to understand Ext groups in this category, and we need to see k-invariants associated to $K_{(p)}$-local spectra as elements in such Ext groups. We outline these steps without proof. The proofs can be easily adapted from those in [2].

1) The category $\mathcal{D}A_*$.

For $i \in \mathbb{Z}$, there is an automorphism $T^i : \mathcal{D}A \to \mathcal{D}A$ with $T^i M$ equal to M as a $\mathbb{Z}_{(p)}$-module but with $\Psi^q : T^i M \to T^i M$ equal to $q^i \Psi^q : M \to M$. An object of $\mathcal{D}A_*$ is a collection of objects $M_n \in \mathcal{D}A$ for $n \in \mathbb{Z}$ together with isomorphisms $u : TM_n \cong M_{n+2}$ in $\mathcal{D}A$ for all n. A morphism $f : M \to N$ in $\mathcal{D}A_*$ is a collection of morphisms $f_n : M_n \to N_n$ in $\mathcal{D}A$ such that $uf_n = f_{n+2}u$ for all $n \in \mathbb{Z}$.

The point of this construction is that $K_*(X; \mathbb{Z}_{(p)})$ is an object of $\mathcal{D}A_*$ for any spectrum X.

2) Ext groups in $\mathcal{D}A_*$.

The category $\mathcal{D}A_*$ has enough injectives, allowing the definition of (bigraded) Ext groups. The groups $\mathrm{Ext}^{s,t}_{\mathcal{D}A_*}(-,-)$ vanish for $s > 2$,

essentially as a consequence of the exact sequence of Theorem 5.7. There is an Adams spectral sequence with

$$E_2^{s,t}(X,Y) = \mathrm{Ext}^{s,t}_{\mathcal{D}A_*}\big(K_{(p)*}(X), K_{(p)*}(Y)\big),$$

converging strongly to $[X_{K_{(p)}}, Y_{K_{(p)}}]_*$.

3) k-invariants and the category $k\mathcal{D}A_*$.

To each $K_{(p)}$-local spectrum X is associated a k-invariant $k_X \in \mathrm{Ext}^{2,1}_{\mathcal{D}A_*}\big(K_{(p)*}(X), K_{(p)*}(X)\big)$. The only non-trivial differential d_2 in the Adams spectral sequence can be expressed in terms of these k-invariants. We form the additive category $k\mathcal{D}A_*$ whose objects are pairs (M,κ), with $M \in \mathcal{D}A_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{D}A_*}(M,M)$, and whose morphisms from (M,κ) to (N,λ) are morphisms $f : M \to N$ in $\mathcal{D}A_*$ with $\lambda f = f\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{D}A_*}(M,N)$.

This now allows us to translate the main result of [2] into our setting.

Theorem 6.1. *Homotopy types of $K_{(p)}$-local spectra are in one-to-one correspondence with isomorphism classes in $k\mathcal{D}A_*$.* □

7. Appendix: The Split Setting

In this section we summarise the Adams summand analogues of our results. We omit proofs as these are easy adaptations of those given in the preceding sections.

For a fixed odd prime p, we denote the periodic Adams summand by G, and we write B for the ring of stable degree zero operations $G^0(G)$. As usual, we choose q primitive modulo p^2, and we set $\hat{q} = q^{p-1}$. Let $\hat{\Phi}_n = \prod_{i=1}^{n}(\Psi^q - \hat{q}^{(-1)^i \lfloor i/2 \rfloor}) \in B$. The elements $\hat{\Phi}_n$, for $n \geqslant 0$, form a topological $\mathbb{Z}_{(p)}$-basis for B; see [5, Theorem 6.13]. Just as for A, the topological ring B can be viewed as a completion of the polynomial ring $\mathbb{Z}_{(p)}[\Psi^q]$. The results in the two cases are formally very similar, differing only in that in many formulas q must be replaced by $\hat{q}$.

Let $\mathcal{D}B$ denote the category of discrete B-modules, defined in the obvious way by analogy with Definition 2.3. There is a corresponding graded version $\mathcal{D}B_*$. Then the additive category $k\mathcal{D}B_*$ is formed just as we did in the non-split setting. Its objects are pairs (M,κ), where $M \in \mathcal{D}B_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{D}B_*}(M,M)$.

Theorem 7.1.

(1) *$\mathcal{D}B$ is a cocomplete abelian category with enough injectives.*
(2) *For any spectrum X, $G_0(X)$ is an object of $\mathcal{D}B$, and $G_*(X)$ is an object of $\mathcal{D}B_*$.*
(3) *$\mathcal{D}B$ is isomorphic to Bousfield's category $\mathcal{B}(p)$.*
(4) *The functor $U(-) = \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Z}_{(p)}}(B,-)$ from $\mathbb{Z}_{(p)}$-modules to $\mathcal{D}B$ is right adjoint to the forgetful functor.*

(5) *For any M in $\mathcal{D}B$, there is an exact sequence in $\mathcal{D}B$*

$$0 \to M \xrightarrow{\alpha} UM \xrightarrow{\beta} UM \xrightarrow{\gamma} M \otimes \mathbb{Q} \to 0.$$

(6) *The groups* $\mathrm{Ext}^{s,t}_{\mathcal{D}B_*}(-,-)$ *vanish for* $s > 2$.

(7) *Homotopy types of G-local spectra are in one-to-one correspondence with isomorphism classes in $k\mathcal{D}B_*$.* □

References

[1] J. F. Adams, Operations of the n^{th} kind in K-theory and what we don't know about $\mathbb{R}P^\infty$, in *New Developments in Topology*, ed. G. Segal, Cambridge University Press, Cambridge, 1974.

[2] A. K. Bousfield, On the homotopy theory of K-local spectra at an odd prime, Amer. J. Math. 107 (1985), 895–932.

[3] A. K. Bousfield, A classification of K-local spectra, J. Pure Appl. Algebra 66 (1990), 121–163.

[4] F. Clarke, M. D. Crossley and S. Whitehouse, Bases for cooperations in K-theory, K-Theory 23 (2001), 237–250.

[5] F. Clarke, M. D. Crossley and S. Whitehouse, Algebras of operations in K-theory, Topology 44 (2005), 151–174.

[6] J. Franke, Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence, K-theory archives 139, 1996.

[7] L. Fuchs, *Infinite Abelian Groups, Volume II*, Academic Press, 1973.

[8] N. Ganter, Smash products of $E(1)$-local spectra at an odd prime, `http://arxiv.org/abs/math.AT/0410589`, 2004.

[9] K. Johnson, The action of the stable operations of complex K-theory on coefficient groups, Illinois J. Math. 28 (1984), 57–63.

[10] K. Johnson, The algebra of stable operations for p-local complex K-theory, Canad. Math. Bull. 30 (1987), 57–62.

[11] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, London, 1968.

[12] C. Roitzheim, On the algebraic classification of K-local spectra, `http://www.math.uni-bonn.de/people/cro/klocal.ps`, 2005.

[13] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, Cambridge, 1994.

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